A Class of Sums with Unexpectedly High Cancellation

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Pentagonal Number Theorem

Let $p(n)$ be the number of partitions of n and $G_\ell = \frac{\ell(3\ell-1)}{2}$ be $\ell-$th pentagonal number. Then

$$\sum_{G_\ell \leq n} (-1)^\ell p(n - G_\ell) = 0.$$

Proof in for example Professor Berndt's "Number Theory in Spirit of Ramanujan" Book.

Rademacher expression for $p(n)$

Let $\mu_k(n) = \frac{\pi\sqrt{24n-1}}{6k}$. Rademacher-Ramanujan-Hardy proved that

$$p(n) = \frac{\sqrt{12}}{24x-1}\left(\sum_{k=1}^{\infty} A_k(n)\left((1-\frac{1}{\mu_k(n)})e^{\mu_k(n)} + (1+\frac{1}{\mu_k(n)})e^{-\mu_k(n)}\right)\right).$$

where

$$A_k(n) = \sum_{\substack{0\leq h<k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi ihn}{k}}.$$

Proof in for example Professor Andrew's "Theory of partitions" book.
Approximate version

$$p(n) \simeq \frac{e^{\pi\sqrt{\frac{2n}{3}}}}{4\sqrt{3n}}.$$

Approximation of number of partitions

The first two terms:

$$p_2(x) = \frac{\sqrt{12}e^{\frac{\pi}{6}\sqrt{24x-1}}}{24x-1}\left(1 - \frac{6}{\pi(24x-1)^{\frac{1}{2}}}\right) + O(p(x)^{0.5}).$$

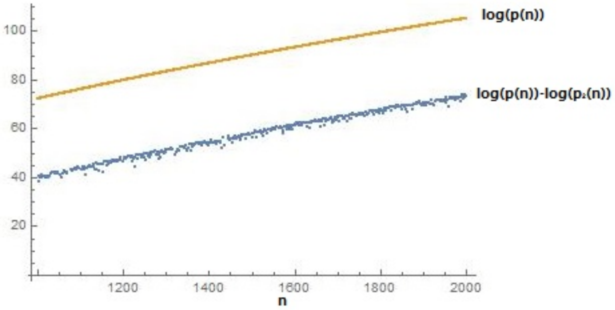

Figure: Comparison of the error term of first two terms with actual number for $20 < n < 2000$.

Conclusion

What will happen if we use $p_2(n)$ in pentagonal number theorem?

$$\sum_{G_\ell \leq n} (-1)^\ell p_2(n - G_\ell) = O(p(n)^{0.5}).$$

Beginning of a long story!

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Observation

Lets test something simpler! We proved that

$$\sum_{l^2 \leq n} (-1)^l e^{\sqrt{n-l^2}} = O(e^{\frac{\sqrt{n}}{100}}).$$

There is something deeper than a combinatorial property.

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Observation

Lets test something simpler!

$$\sum_{\ell^2 \leq n} (-1)^\ell e^{\sqrt{n-\ell^2}} = O(n^{10}).$$

Our estimated error term is very small!

Theorem

Let $b, d \in \mathbb{R}$, $a, c > 0$; *Also, let* $h(x)$ *be* $(\alpha x + \beta)^t$ *for* $\alpha, \beta, t \in \mathbb{R}$. *Then*

$$\sum_{n: an^2+bn+d<x} (-1)^n e^{c\sqrt{x-(an^2+bn+d)}} h(x - (an^2 + bn + d)) \ll \sqrt{x} e^{wc\sqrt{x}}. \tag{1}$$

where $w > 0$ *is defined as follows. Set*

$$\Delta(r) := \sqrt{\sqrt{a} r \frac{\sqrt{ar^2 + 4} + r\sqrt{a}}{2} - \frac{\pi r}{c}} \quad , \quad r \geq 0$$

If $r = \alpha$ *is when* $\Delta(r)$ *is maximized, then* $w = \min(1, \Delta(\alpha))$.

A heuristic argument

Consider Bernoulli random variables $\epsilon_n = \pm 1$ with probability $P(\epsilon_n = 1) = 0.5$. Then what is expectation of

$$E\left(\sum_{\ell^2<n} \epsilon_\ell e^{\sqrt{n-\ell^2}}\right) = 0$$

$$Var\left(\sum_{\ell^2<n} \epsilon_\ell e^{\sqrt{n-\ell^2}}\right) \gg e^{2\sqrt{n}}.$$

Then why this sum is that small?

First Natural try

Understanding using Taylor expansions:

$$\sum_{r=0}^{\infty} \frac{S_r(M)}{r!} := \sum_{r=0}^{\infty} \sum_{\ell^2 \leq 4M^2} (-1)^{\ell} \frac{(4M^2 - \ell^2)^{\frac{r}{2}}}{r!} = O(e^{\frac{2M}{50}}).$$

We expect that $\deg(S_r(M)) = 2r$.

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Reality

We proved that actually $\deg(S_r(M)) = r - 1$.

$$S_4(M) = 16M^3 - 17M$$
$$S_6(M) = -408M^5 - 480M^3 - 2073M.$$

This might be doable by using the known result about Bernoulli numbers. We did not attempt to prove it this way; we predict that this way can prove this polynomial case in the best case scenario.

Remark

These sums are similar to Kloosterman's sum, I call them sisters! Our interested series:

$$\sum_{\ell^2 \leq n} (-1)^\ell e^{\sqrt{n-\ell^2}}$$

Kloosterman's sum:

$$\sum_{\substack{0 \leq \ell < n \\ \gcd(n,\ell)=1}} e^{2\pi i \ell^{-1}(a-\ell^2)}$$

We prove a weaker result about their father:

$$\sum_{\ell^2 < Tx} (-1)^\ell e^{(\alpha+i\beta)\sqrt{x-\frac{\ell^2}{T}}} = O\left(\sqrt{\frac{Tx}{|\beta|+1}} e^{\alpha(\sqrt{\frac{2}{2+\pi^2}}+\delta)\sqrt{x}} + \sqrt{T}\right).$$

Theorem

Assume $\epsilon > 0$, x *is large enough and* $a = 1 - \sqrt{\frac{2}{2+\pi^2}}$. *We have*

$$\sum_{\ell^2 < xe^{\frac{4a}{3}\sqrt{x}}} (-1)^\ell \Psi\left(e^{\sqrt{x - \ell^2 e^{-\frac{2a}{3}\sqrt{x}}}}\right) = O\left(e^{(1-\frac{a}{3}+\epsilon)\sqrt{x}}\right).$$

Remarks

In particular for $T = e^{0.786\sqrt{x}}$:

$$\sum_{0<2\ell<\sqrt{xT}} \Psi([e^{\sqrt{x-\frac{(2\ell)^2}{T}}},\ e^{\sqrt{x-\frac{(2\ell-1)^2}{T}}}]) = \Psi(e^{\sqrt{x}})\left(\frac{1}{2} + O\left(e^{-0.196\sqrt{x}}\right)\right),$$

This says we have half of primes in $\cup[e^{\sqrt{x-\frac{(2\ell)^2}{T}}},\ e^{\sqrt{x-\frac{(2\ell-1)^2}{T}}}]$.
If we use RH naively, we cannot make the error term this small. So controlling the father can be really rewarding!!!

General case

In general if we consider a "Meinardus type" partition as

$$\lambda(n) \sim (g(n))^q e^{(k(n))^\theta} \left(1 - \frac{1}{(h(n))^r}\right) + O(\lambda(n)^s)$$

where $0 < s < 1$ and $\theta, r, q > 0$ and $k(n)$ is a linear polynomial and $g(n), h(n)$ are rational functions. Then

$$\sum_{t(\ell)<n} (-1)^\ell \lambda(n - t(\ell)) = O(\lambda^\kappa(n))$$

where $\kappa > s$ is determined from the analytic properties of the partition.

Application in Partitions

We proved that for the usual partitions

$$\sum_{\ell^2 < x} (-1)^\ell p(x - \ell^2) \ \sim \ 2^{-3/4} x^{-1/4} \sqrt{p(x)}.$$

and for the partitions with distinct parts

$$\sum_{\ell^2 \leq n} (-1)^\ell q(n - \ell^2) = O(\sqrt[3]{q(n)})$$

Application to Prouhet-Tarry-Escott problem

The problem asks for biggest possible $k < n$ such that there exists two disjoint sets $\{y_i\}$ and $\{x_j\}$ such that

$$x_1 + x_2 + \cdots + x_n = y_1 + y_2 + \cdots + y_n$$
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For the ideal solution - $k = n - 1$ - the best we know is $n = 12$.

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For the non-constructive solution, the best we know is $k = O(\sqrt{n})$ using a certain Peigen-hole argument.
We can suggest a solution to an "approximate" version of PTE problem.

Theorem

Let $n \geq 1$ and define $N = \lfloor (2n)^{\frac{2}{3}} \rfloor$. Let for $1 \leq i \leq n$

$$x_i = N^3 - (2i-2)^2 \in \mathbb{N} \qquad y_i = N^3 - (2i-1)^2 \in \mathbb{N} \qquad .$$

Then for all $1 \leq r \ll k := \frac{n^{\frac{2}{3}}}{\log(n)}$ we have

$$\sum_{1 \leq i \leq n} x_i^r - \sum_{1 \leq i \leq n} y_i^r = O\left((\max(x_i, y_i))^{\frac{5r}{6}} \right) = O(N^{\frac{5r}{2}}). \qquad (2)$$

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So modulo error $O\left((\max(x_i, y_i))^{\frac{5r}{6}}\right)$ *we have* $k \gg n^{\frac{2}{3}+\epsilon}$.

Proof using circle method

We just talk about the case $\sum_{\ell^2 \le n} (-1)^\ell e^{\sqrt{n-\ell^2}}$. Let

$$f(z) = \frac{e^{\sqrt{n-z^2}}}{\sin(\pi z)}$$

and find $\int_\gamma f(z)dz$.
By the Residue Theorem the residue will be exactly the sum we are interested.

$$\int_\gamma f(z)dz = \sum_{\ell^2 \le n} (-1)^\ell e^{\sqrt{n-\ell^2}}.$$

We need to compute the integral over contour.

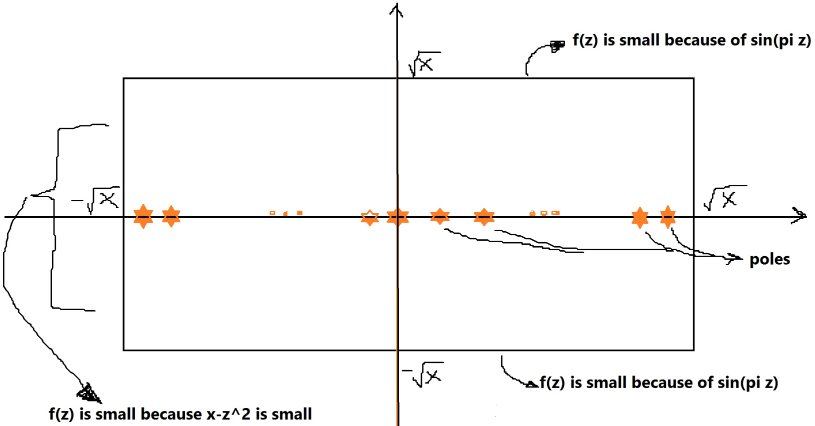

Figure: Contour γ

Thank You

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