Pentagonal Number Theorem and Riemann Hypothesis

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Analytic and Combinatorial Number Theory, The Legacy of Ramanujan
In honor of Professor Berndt's 80th birthday
University of Illinois at Urbana Champaign, Urbana Champaign , Illinois

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Outline

1. Pentagonal Number Theorem
2. The Conjecture
3. The Conjecture vs Riemann Hypothesis

Pentagonal Number Theorem

Let $p(n)$ be the number of partitions and $G_l = \frac{l(3l-1)}{2}$ be $l-$th pentagonal number. Then

$$\sum_{G_l \le n} (-1)^l p(n - G_l) = 0.$$

Sketch of proof (Professor Berndt's "Number Theory in Spirit of Ramanujan" Book)

1. Elliptic discussion: Finding coefficients of $(z;q)_\infty = \sum_n B_n(a,q) z^n$.
2. Solving recurrence formula $B_n = f(a,q) B_{n-1}$.
3. Literally taking $a \longrightarrow \infty$ and changing z properly.
4. Concluding $(q;q)_\infty = \sum_{l=-\infty}^{\infty} (-1)^l q^{G_l}$.
5. Considering $(q;q)_\infty^{-1} = \sum_{n \ge 0} p(n) q^n$.

Rademacher expression for $p(n)$

Let $\mu_k(n) = \frac{\pi\sqrt{24n-1}}{6k}$. Rademacher-Ramanujan-Hardy proved that

$$p(n) = \frac{\sqrt{12}}{24x-1}\left(\sum_{k=1}^{\infty} A_k(n)\left((1-\frac{1}{\mu_k(n)})e^{\mu_k(n)} + (1+\frac{1}{\mu_k(n)})e^{-\mu_k(n)}\right)\right).$$

where

$$A_k(n) = \sum_{\substack{0\le h<k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi ihn}{k}}.$$

Proof in Professor Andrew's "Theory of partitions" book.

1. Cauchy integral formula
2. Farey dissections to avoid singularities
3. Modularity of Generating function
4. Circle Method

Generalization ...

Define

$$p(x) = \frac{\sqrt{12}}{24x-1}\left(\sum_{k=1}^{\infty} A_k(x)\left((1-\frac{1}{\mu_k(x)})e^{\mu_k(x)} + (1+\frac{1}{\mu_k(x)})e^{-\mu_k(x)}\right)\right).$$

where

$$A_k(x) = \sum_{\substack{0\le h<k \\ (h,k)=1}} \omega_{h,k} e^{\frac{2\pi i h[x]}{k}}.$$

Intuition

The first two terms:

$$p(x) = \frac{\sqrt{12}e^{\frac{\pi}{6}\sqrt{24x-1}}}{24x-1}\left(1 - \frac{6}{\pi(24x-1)^{\frac{3}{2}}}\right) + O(p(x)^{0.5}).$$

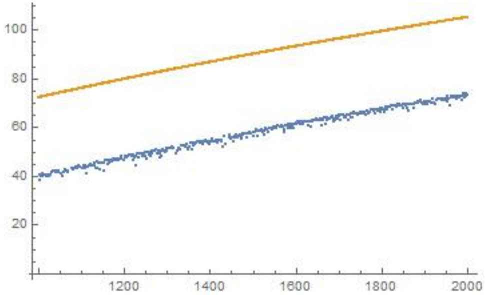

Figure: Comparison of the error term of first two terms with actual number for $20 < n < 2000$.

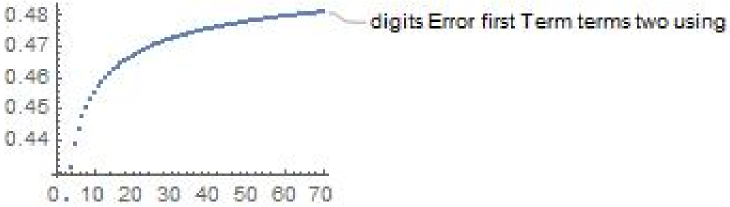

Figure: Relative error of the first two terms for $500 < x < 70000$.

Why generalization?

Let $p_1(x)$ be the first term of the Rademacher formula. Then

$$\sum_{G_l < x} p_1(x - G_l)(-1)^l = O(p(x)^{0.72}).$$

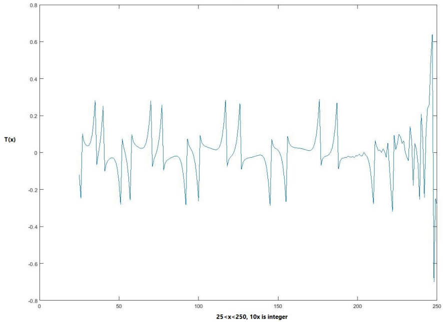

Figure: Error term in Pentagonal Number Theorem for $20 < n < 250$

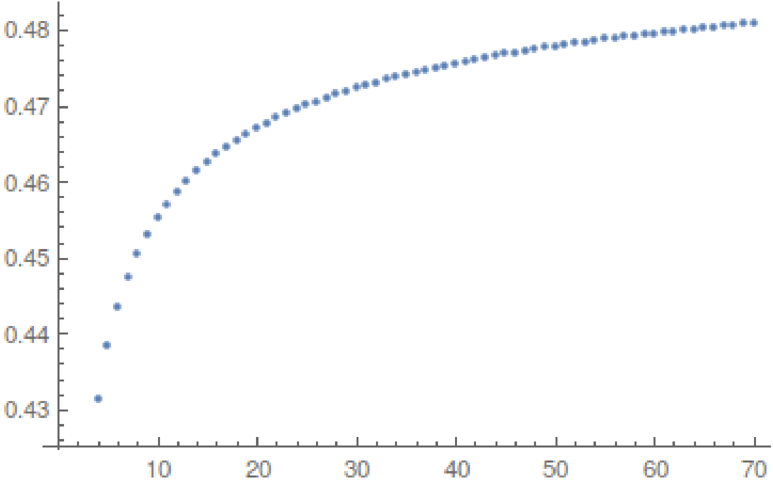

Figure: Error term in Pentagonal Number Theorem for $500 < n < 70000$

Unexpected issue in natural proof!

The main difficulty is that we cannot use Cauchy integral formula. i. e.

$$p(x) \neq \int_C \frac{P(q)dq}{q^{x+1}}$$

1. The output of integral is not equal to the Rademacher formula numerically!
2. But they are comparable even for large numbers! Possibly for $A_k(x)$?!
3. **You cannot use Taylor series hoping to get an error better than $\frac{p(x)}{polynomial}$.**
4. Picking a proper function and a contour and use Residue Theorem.

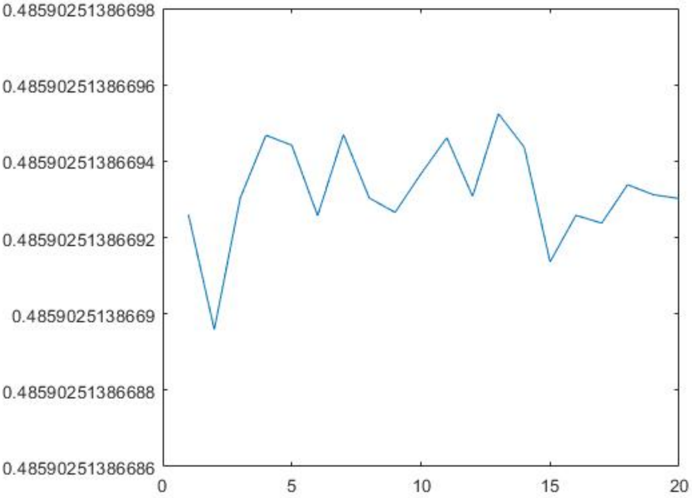

Figure: Relative error $\frac{log(integral)-\log(p(x))}{\log(p(x))}$ which should be around 0.5.

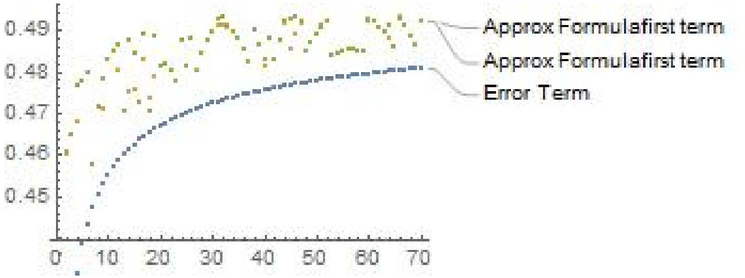

Figure: Relative error of pentagonal number theory for first and second term and the relative error of fisrt two terms.

The most general form so far

Let $r \in \mathbb{N}$, $b, d \in \mathbb{R}$, $c > 0$, and let $0 \leq w < 1$ be defined as follows.

$$w = \inf\{0 < w' < 1 : \frac{-cw' r\pi + c\sqrt{r^2\pi^2 - c^2 + c^2 w'^2}}{r^2\pi^2 - c^2} < \frac{w'^2}{\sqrt{1 + w'^2}}\} \tag{1}$$

Also, let $\xi_r = e^{\frac{\pi i}{r}}$ and $h(x)$ be a polynomial of $\frac{1}{\sqrt{x}}$ with real coefficients. Then

$$\sum_{n:\, \frac{n^2+bnr+dr^2}{r^2} < x} \xi_r^n \frac{e^{c\sqrt{x - \frac{n^2+bnr+dr^2}{r^2}}}}{h(\frac{n}{r})} = O\left(e^{cw\sqrt{x}}\right). \tag{2}$$

What we may expect at first?

One can expect

$$\sum_{G_l < x} (-1)^l p(x - G_l) = O(\frac{p(x)}{x^a}).$$

for some a. Why? If $x_i = e^{y_i}$. then

1. Let $x_1, x_2, \cdots, x_n \in \mathbb{R}$. Let $d_1, d_2, \cdots, d_n$ be ± 1 with equal probability.
2. Obviously $E(d_1 x_1 + d_2 x_2 + \cdots + d_n x_n) = 0$.
3. Again obviously $Var(d_1 x_1 + d_2 x_2 + \cdots + d_n x_n) = x_1^2 + \cdots + x_n^2$.
4. Standard argument suggests $d_1 x_1 + d_2 x_2 + \cdots + d_n x_n$ can be around $\sqrt{x_1^2 + \cdots + x_n^2}$.

So we have a very high level (miraculous!) of cancellation here!

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2. The Conjecture
3. The Conjecture vs Riemann Hypothesis

Comparing $\Psi(e^{\frac{\pi}{6}\sqrt{24x-1}})$ and $p(x)$

Let Ψ be Chebyshev function. Assuming Riemann Hypothesis and Pentagonal number theorem:

$$\sum_{G_l<x}(-1)^l\Psi(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$
$$=O\left(\frac{e^{\frac{0.72\pi}{6}\sqrt{24x-1}}}{\sqrt{24x-1}}\right)$$

Question: For the above formula, we used the estimation $\Psi(x)=x+\theta(x^{\frac{1}{2}+\delta})$. What if we use the exact amount of Ψ?

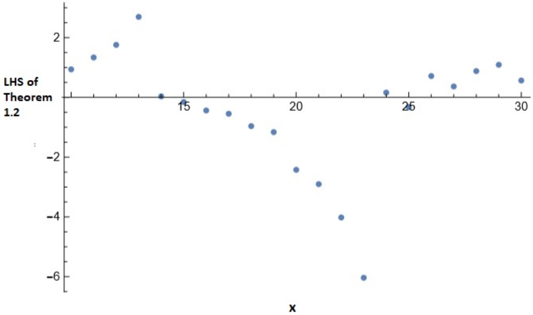

Figure: Error term of the theorem for Chebyshev Ψ function for $10 < n < 30$

Observation

This nice behaviour of Chebyshev functions means something deep is going on here; which motivates us to assume the following hypothesis.

The conjecture: weak version

$$\sum_{G_l<x}\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)(-1)^l\sum_{n\leq e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}}}\frac{\Lambda(n)}{n^s}$$
$$=O\left(e^{\frac{\pi(\frac{1}{2}-\sigma+\delta)}{6}\sqrt{24x-1}}\right).$$

A dangerous intuition

Watching the error terms in partitions, we thought it remains really small like $O(1)$. But it started exploding to the expected error after $x=400$. So the same may happen for Ψ. Unfortunatly we do not have the technology to check it.

Theorem

Assuming the error term of pentagonal number theorem is $p(x)^{0.5}$, *then*

1. *For case* $Re(s) = 0$, *for* $\delta > 0$, *and a.e.* t

$$\sum_{G_l<x}(-1)^l\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right) \times \sum_{n\le e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}}}\frac{\Lambda(n)}{n^{it}} = O\left(e^{\frac{\pi\delta}{6}\sqrt{24x-1}}\right).$$

2. *For case* $Re(s) = \frac{1}{2}$, *for* $\delta > 0$, *and a.e.* t

$$\sum_{G_l<x}(-1)^l\left(\frac{1}{24(x-G_l)-1}-\frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right) \times \sum_{n\le e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}}}\frac{\Lambda(n)}{n^{\frac{1}{2}+it}} = O\left(e^{\frac{\pi(\frac{1}{4}+\delta)}{6}\sqrt{24x-1}}\right).$$

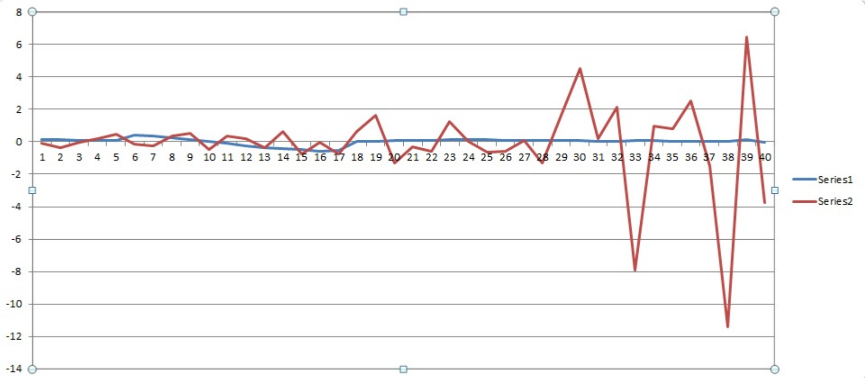

Figure: Error term of the right hand side of hypothesis for $s = 0$ and $s = \frac{1}{2}$.

The Conjecture: strong version

Let $\epsilon > 0$ and $s = \sigma + it \in \mathbb{C}$. Then there exists a function f with finite Fourier Transform such that

$$\sum_{G_l < x + \frac{1}{24} + \epsilon} (-1)^l \left(\frac{1}{24(x - G_l) - 1} - \frac{6}{\pi(24(x - G_l) - 1)^{\frac{3}{2}}} \right) \times \sum_{n \le e^{\frac{\pi}{6}\sqrt{24(x - G_l) - 1}}} \frac{\Lambda(n)}{n^s} = O\left(f(x) e^{\frac{\pi(\frac{1}{2} - \sigma)}{6}\sqrt{24x - 1}} \right).$$

Another Conjecture

Consider distribution of a random sequence $\{\rho_m\}$ as follows

$$\sum_{0<Im(\rho_m)<T} 1 = \frac{T}{2\pi}\log(\frac{T}{2\pi}) - \frac{T}{2\pi} + O(\log(T)).$$

Then the function that minimize

$$\sum_{G_l<x}(-1)^l\omega(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1} - \frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$

is chebyshev Ψ function. In particular

$$\sum_{G_l<x}(-1)^l\Psi(e^{\frac{\pi}{6}\sqrt{24(x-G_l)-1}})\left(\frac{1}{24(x-G_l)-1} - \frac{6}{\pi(24(x-G_l)-1)^{\frac{3}{2}}}\right)$$
$$= O(f(x)).$$

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Theorem

The strong version of the hypothesis for $\sigma = \frac{1}{2}$ results in Riemann hypothesis.

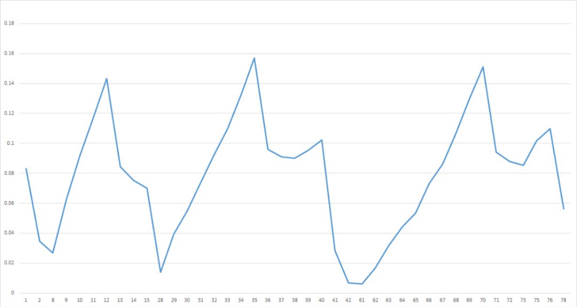

Figure: RHS of strong Conjecture for $1 < n < 90$ after one week of running using parallel programming.

Thank You

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HAPPY BIRTHDAY PROFESSOR BERNDT

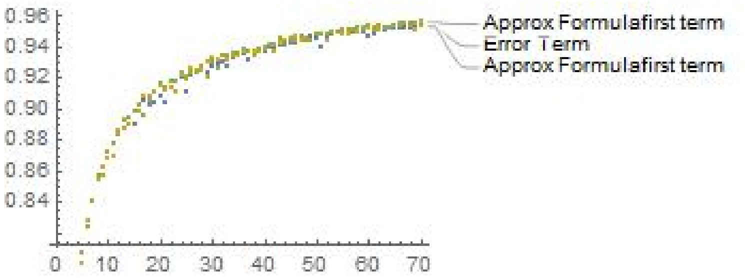

Figure: Relative error of pentagonal number theory for first and second term and the relative error of fisrt two terms with 50 digits.